Twisted configurations over quantum Euclidean spheres

Giovanni Landi [a,*], John Madore [b,c]

[a] *Dipartimento di Scienze Matematiche, Università di Trieste, Via Valerio 12/b, I-34127 Trieste, Italy*
[b] *Laboratoire de Physique Théorique, Université de Paris-Sud, Bâtiment 211, F-91405 Orsay, Germany*
[c] *Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 München, Germany*

Abstract

We show that the relations which define the algebras of the quantum Euclidean planes $\mathbb{R}_q^N$ can be expressed in terms of projections provided that the unique central element, the radial distance from the origin, is fixed. The resulting reduced algebras without center are the quantum Euclidean spheres S_q^{N-1}. The projections $e = e^2 = e^*$ are elements in $\mathrm{Mat}_{2^n}(S_q^{N-1})$, with $N = 2n + 1$ or $N = 2n$, and can be regarded as defining modules of sections of q-generalizations of monopoles, instantons or more general twisted bundles over the spheres. We also give the algebraic definition of normal and cotangent bundles over the spheres in terms of canonically defined projections in $\mathrm{Mat}_N(S_q^{N-1})$.
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1. Introduction

Let $\mathcal{A}$ be an associative algebra defined formally in terms of a set of N generators x^i and a set of relations $R(x^i)$. Suppose that for a fixed value of each of the parameters which enter in the definition of the $R(x^i)$ some of the relations reduce to the constraint that the algebra be commutative. We refer to these as commutation relations. The remaining relations become constraints in the commutative limit and define the noncommutative generalization of a

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* Corresponding author.
E-mail address: landi@dsm.univ.trieste.it (G. Landi).

submanifold of $\mathbb{R}^N$. We suppose for simplicity that there is only one relation of this form. The commutative limit will describe then a submanifold V of dimension $N - 1$ embedded in $\mathbb{R}^N$. If one introduce the moving frame $\mathrm{d}x^i$ on $\mathbb{R}^N$ then the latter acquires the structure of a flat differential manifold. This defines by the embedding, i.e. by the relations, a moving frame θ^α on V, which one can always choose so that locally the last component is normal to V. When V is parallelizable this choice can be made globally and the module of sections $\mathcal{T}$ of the cotangent bundle $T^*(V)$ is free. In all cases the embedding defines a splitting of the module of sections $\mathcal{A}^N$ of $T^*(\mathbb{R}^N)$ into a direct sum

$$\mathcal{A}^N = \mathcal{T} \oplus \mathcal{N} \tag{1.1}$$

of $\mathcal{T}$ and a free module $\mathcal{N}$ of rank 1. The metric and the frame on V are determined by the embedding relations. The construction is most elegantly described in terms of projections in the matrix algebra $\mathrm{Mat}_N(\mathcal{A})$, i.e. elements $e \in \mathrm{Mat}_N(\mathcal{A})$ such that $e^2 = e = e^*$. It is a consequence of the Serre-Swan theorem (cf. [7,15,18,20]) that to every vector bundle over V corresponds an equivalence class of projections in a matrix algebra $\mathrm{Mat}_r(\mathcal{A})$, for a suitable r. The module of sections of the bundle is a projective module of finite type over the algebra $\mathcal{A}$ and conversely any such a module can be realized as the module of sections of a bundle; and these modules are naturally characterized by projections. This correspondence quite naturally generalizes the notion of a vector bundle over a noncommutative algebra.

We shall see that in the case of the quantum Euclidean spheres S_q^{N-1} a projection $e \in \mathrm{Mat}_{2^n}(S_q^{N-1})$, with $N = 2n$ or $N = 2n + 1$, can be so chosen that the relations which define it ($e^2 = e = e^*$) are equivalent to the relations $R(x^i)$ which define the algebra $\mathcal{A}$ of the spheres. As already mentioned, the projections can be regarded as defining modules of sections of bundles over the spheres which then will be q-generalizations of monopoles, instantons or more general twisted configurations. As in the classical situation, the bundles should be characterized by integer valued topological charges (Chern numbers); work on these is in progress and will be reported somewhere else.

In Section 2, some basic results [13] about the general N-dimensional quantum Euclidean spaces $\mathbb{R}_q^N$ and quantum Euclidean spheres S_q^{N-1} are reviewed. In Section 3, we shall introduce the projections which determine (and are determined by) the spheres relations and which, in turn, define twisted configurations over the spheres S_q^{N-1}. We shall describe in detail the cases $N = 3, 4, 5, 6$, i.e. the spheres of dimensions 2, 3, 4 and 5, respectively, and outline the general case. Not surprisingly, 'even- and odd-dimensional' spheres will behave differently. In Section 4, we construct the projections for the normal and cotangent bundles over the spheres S_q^{N-1}. The final remarks concern some preliminary results on the computation of topological charges.

2. Quantum Euclidean planes and spheres

In this section some basic results [13] (see also [6]) about the general N-dimensional quantum Euclidean spaces $\mathbb{C}_q^N$ and $\mathbb{R}_q^N$ and spheres S_q^{N-1} are reviewed. We start with the matrix $\hat{R}$ for the quantum group $\mathrm{SO}_q(N, \mathbb{C})$. It is a symmetric $N^2 \times N^2$ matrix and its main

property is that it satisfies the braid relation. It admits a projector decomposition:

$$\hat{R} = qP_{(s)} - q^{-1}P_{(a)} + q^{1-N}P_{(t)},   \tag{2.1}$$

where the $P_{(s)}$, $P_{(a)}$, $P_{(t)}$ are $SO_q(N)$-covariant q-deformations of the symmetric trace-free, antisymmetric and trace projectors, respectively. The projector $P_{(t)}$ projects onto a one-dimensional subspace and can be written in the form $P_{(t)}{}^{ij}{}_{kl} = (g^{mn}g_{mn})^{-1}g^{ij}g_{kl}$. This leads to the definition of a deformed metric matrix, the $N \times N$ matrix g_{ij} given by

$$g_{ij} = q^{-\rho_i}\delta_{i,-j}.   \tag{2.2}$$

If n is the rank of $SO(N, \mathbb{C})$, i.e. the integer part in $N/2$, the indices take the values $i = -n, \ldots, -1, 0, 1, \ldots, n$ for $N = 2n + 1$, and $i = -n, \ldots, -1, 1, \ldots, n$ for $N = 2n$. Moreover, we have introduced the notation

$$\rho_i = \begin{cases} (n - \tfrac{1}{2}, \ldots, \tfrac{1}{2}, 0, -\tfrac{1}{2}, \ldots, \tfrac{1}{2} - n) & \text{for } N = 2n + 1, \\ (n - 1, \ldots, 0, 0, \ldots, 1 - n) & \text{for } N = 2n. \end{cases}   \tag{2.3}$$

The metric and the braid matrix satisfy the 'gTT' relations [13]

$$g_{il}\hat{R}^{\pm 1 lh}{}_{jk} = \hat{R}^{\mp 1 hl}{}_{ij}g_{lk}, \qquad g^{il}\hat{R}^{\pm 1 jk}{}_{lh} = \hat{R}^{\mp 1 ij}{}_{hl}g^{lk}.   \tag{2.4}$$

With the help of the projector $P_{(a)}$, the N-dimensional quantum Euclidean space is defined as the associative algebra $\mathbb{C}_q^N$ generated by elements $\{x^i\}_{i=-n,\ldots,n}$ with relations

$$P_{(a)}{}^{ij}{}_{kl}x^k x^l = 0.   \tag{2.5}$$

or, more explicitly [23]

$$\begin{aligned} x^i x^j &= qx^j x^i \quad \text{for } i < j, i \neq -j, \\ [x^i, x^{-i}] &= \begin{cases} k\omega_{i-1}^{-1}r_{i-1}^2 & \text{for } i > 1, \\ 0 & \text{for } i = 1, \quad N = 2n, \\ hr_0^2 & \text{for } = 1, \quad N = 2n + 1. \end{cases} \end{aligned}   \tag{2.6}$$

We use the notation $\omega_i = q^{\rho_i} + q^{-\rho_i}$, $h = q^{1/2} - q^{-1/2}$, $k = q - q^{-1}$ and

$$r_i^2 = \sum_{k,l=-i}^{i} g_{kl}x^k x^l, \qquad i \geq 0 \quad \text{for } N = 2n + 1, \quad i \geq 1 \text{ for } N = n.   \tag{2.7}$$

The last element $r^2 \equiv r_n^2$ can be shown to be central.

For $q \in \mathbb{R}^+$ a conjugation $(x^i)^* = x^j g_{ji}$ can be defined on $\mathbb{C}_q^N$ to obtain what is known as the quantum real Euclidean space $\mathbb{R}_q^N$. The relations (2.6) can be used to derive analogous

ones for the variables $\{(x^i)^*, x^j\}$,

$$x^i x^j = q x^j x^i \quad \text{for } i < j, \qquad x_i^* x_j = q x_j x_i^*, \quad i \neq j,$$

$$[x^i, (x^i)^*] = \begin{cases} \left[\dfrac{(1-q^{-2})}{(1+q^{-2\rho_{i-1}})} \right] r_{i-1}^2 & \text{for } i > 1, \\ 0 & \text{for } i = 1, \ N = 2n, \\ (1-q^{-1}) r_0^2 & \text{for } i = 1, \ N = 2n+1. \end{cases} \qquad (2.8)$$

The central element r^2 can be written as

$$r^2 = \sum_{k=-n}^{n} (x^k)^* x^k = q^{-2\rho_n} x^n (x^n)^* + \cdots + (x^n)^* x^n, \qquad (2.9)$$

with similar expressions for the elements r_i^2, $i \geq 0$ for $N = 2n+1$, $i \geq 1$ for $N = n$,

$$r_i^2 = \sum_{k=-i}^{i} (x^k)^* x^k = q^{-2\rho_i} x^i (x^i)^* + \cdots + (x^i)^* x^i. \qquad (2.10)$$

By fixing the value of r^2 we get the quantum Euclidean sphere S_q^{N-1} of the corresponding radius. Thus, the quantum Euclidean sphere S_q^{N-1} is naturally considered as a quantum subspace of the quantum Euclidean space $\mathbb{R}_q^N$ and the algebra of functions on the sphere is a quotient of the algebra of functions on the quantum Euclidean space, the quotient being taken by the ideal generated by the relation that fixes the radius. Furthermore, the spheres S_q^{N-1} are invariant under the action of the quantum groups $SO_q(N)$ on them.

It is easy to see that the spheres S_q^{N-1} have a S^1 worth of classical points. Indeed, let $r \in \mathbb{R} \setminus \{0\}$ be the radius of the sphere. Then, with $\lambda \in \mathbb{C}$ such that $|\lambda|^2 = 1$, there is a family of one-dimensional representations (characters) of the algebra S_q^{N-1} given by

$$\tau_\lambda(1) = 1, \qquad \tau_\lambda(x^n) = \frac{r}{\sqrt{1+q^{-2\rho_n}}} \lambda, \qquad \tau_\lambda((x^n)^*) = \frac{r}{\sqrt{1+q^{-2\rho_n}}} \bar{\lambda},$$

$$\tau_\lambda(x^i) = \tau_\lambda((x^i)^*) = 0, \qquad (2.11)$$

for $i = 0, 1, \ldots, n-1$ or $i = 1, \ldots, n-1$ according to whether $N = 2n+1$ or $N = 2n$. Clearly these representations will yield traces on the algebras S_q^{N-1}.

3. Twisted configurations over S_q^{N-1}

We shall now introduce a hermitian idempotent or projection, an element $e \in \mathrm{Mat}_{2^n}(S_q^{N-1})$ satisfying the conditions $e = e^2 = e^*$ which determine the structure of the algebra S_q^{N-1}. The star here is the formal adjoint in the complete matrix algebra and N is even or odd. These projections, in turn, can be thought of as defining modules of sections of bundles over the spheres S_q^{N-1} which are q-generalizations of monopoles and instanton bundles or more general twisted configurations. We shall use a matrix trace together with the trace

determined by the representations (2.11) to compute the rank which, for the projector in question, will turn out to be equal to 2^{n-1}. At the moment we are unable to compute the topological charges in general; work on this is in progress and will be reported elsewhere. We shall describe in detail the cases $N = 3, 4, 5, 6$, that is the spheres of dimensions 2, 3, 4 and 5, respectively. We shall also outline the general case which in principle can be obtained using the same techniques.

3.1. Monopoles on the Euclidean sphere S_q^2

With a suitable rescaling of the generators the Euclidean sphere S_q^2 can be identified with the so called equator sphere of Podleś [24]. For its presentation, we have the following generators:

$$x^i = (\sqrt{q}x_1^*, x_0, x_1), \tag{3.1}$$

with $(x_0)^* = x_0$, and commutation relations (2.8) given by

$$x_0 x_1 = q x_1 x_0, \qquad x_1^* x_0 = q x_0 x_1^*, \qquad [x_1, x_1^*] = (1 - q^{-1})x_0^2. \tag{3.2}$$

These commutation relations give for the central element r^2 the equivalent expressions

$$r^2 = q x_1 x_1^* + x_0^2 + x_1^* x_1 = q x_0^2 + (1 + q)x_1^* x_1 = q^{-1}x_0^2 + (1 + q)x_1 x_1^*. \tag{3.3}$$

Then, a straightforward computation yields that the element $e \in \text{Mat}_2(S_q^2)$ given by

$$e = \frac{1}{2} \begin{pmatrix} 1 + q^{-1/2}r^{-1}x_0 & (1 + q)^{1/2}r^{-1}x_1 \\ (1 + q)^{1/2}r^{-1}x_1^* & 1 - q^{1/2}r^{-1}x_0 \end{pmatrix} \tag{3.4}$$

which is hermitian by construction, is also idempotent, $e^2 = e$, if and only if all the relations (3.2) and (3.3) which define S_q^2 are satisfied.

When the Euclidean sphere S_q^2 is identified with Podleś equator sphere the projector (3.4) coincides with the one found in [5] where projectors for all Podleś spheres were constructed (a projector on the so called Podleś standard sphere had already been constructed in [16]). The projection (3.4) (and the corresponding vector bundle over S_q^2) is of rank 1,

$$\text{rank}(e) =: \tau_\lambda \circ \text{Tr}(e) = \tau_\lambda \left(1 + \frac{1}{2r}q^{-1/2}(1 - q)x_0 \right) = 1, \tag{3.5}$$

where we have used the one-dimensional representations (2.11) and Tr denotes a matrix trace. In order to compute the topological charge of the bundle we need a cyclic 0-cocycle, i.e. a trace τ^1, on the reduced algebra $\bar{S}_q^2 = S_q^2 / \mathbb{C}1$. This computation is an example of the pairing between K-theory and cyclic cohomology [7]. In the present situation the cyclic cocycle one needs is of degree zero since for S_q^2 all cyclic cohomology is in $HC^0(S_q^2)$ while the even cyclic cohomology $HC^n(S_q^2)$, $n > 0$, is the image of the periodicity operator [22]. The trace τ^1 was found in [22] and on the generators (3.1) gives in particular

$$\tau^1(x_0) = -2rq^{1/2}\frac{1}{1 - q}. \tag{3.6}$$

Then, with $[\mathrm{Tr}(e)] \in \bar{S}_q^2$, the topological charge of the projection e is found to be

$$\tau^1([\mathrm{Tr}(e)]) = \tau^1 \left(\frac{1}{2r} q^{-1/2}(1-q)x_0 \right) = -1, \tag{3.7}$$

as it should be given the analogous computations in [4,16] for the computations of topological charges of q-monopoles on Podleś spheres. The fact that the pairing in (3.7) gives an integer number is a consequence of a noncommutative index theorem [7] since the trace τ^1 is the character of a Fredholm module [22]. The trace (3.6) is singular in the classical limit $q = 1$. In the latter limit the bundle corresponding to the projection (3.4) is the monopole bundle over the classical sphere S^2 of topological charge (first Chern number) equal to -1 which is computed by integrating a 2-form on S^2 [19] (classically the relevant cyclic cocycle is in degree 2 and is associated with the volume form of the sphere).

3.2. Instantons on the Euclidean sphere S_q^4

In this section, we generalize the projector (3.4) on the sphere S_q^2 to a projector on the sphere S_q^4 and indicate how to further generalize it to any even quantum Euclidean sphere. The generators of the algebra S_q^4 are the five elements

$$x^i = (q^{3/2}x_2^*, q^{1/2}x_1^*, x_0, x_1, x_2), \tag{3.8}$$

with $(x_0)^* = x_0$. Then the commutation relations (2.8) are

$$x_i x_j = q x_j x_i, \quad i < j, \qquad x_i^* x_j = q x_j x_i^*, \quad i \neq j, \qquad [x_1, x_1^*] = (1 - q^{-1})x_0^2,$$
$$[x_2, x_2^*] = q^{-1}(1 - q^{-1})(q x_1 x_1^* + x_0^2 + x_1^* x_1) = q^{-1}(1 - q^{-1})(q x_0^2 + (1 + q)x_1^* x_1)$$
$$= q^{-1}(1 - q^{-1})(q^{-1}x_0^2 + (1 + q)x_1 x_1^*). \tag{3.9}$$

These commutation relations give for the central element r^2 the equivalent expressions

$$r^2 = q^3 x_2 x_2^* + q x_1 x_1^* + x_0^2 + x_1^* x_1 + x_2^* x_2 = (1 + q^3)x_2^* x_2 + (1 + q^3)x_1^* x_1$$
$$+ q \frac{1 + q^3}{1 + q} x_0^2 = (1 + q^3)x_2 x_2^* + q^{-2}(1 + q^3)x_1 x_1^* + q^{-3} \frac{1 + q^3}{1 + q} x_0^2. \tag{3.10}$$

To alleviate the notation we now consider the case $r^2 = [(1 + q^3)/(1 + q)] \cdot 1$. Equivalently we could have suitably rescaled the generators. The previous sphere relations reduce to

$$1 = (1 + q)x_2^* x_2 + (1 + q)x_1^* x_1 + q x_0^2 = (1 + q)x_2 x_2^* + q^{-2}[(1 + q)x_1 x_1^* + q^{-1}x_0^2]. \tag{3.11}$$

Then, a straightforward computation yields that the element $e \in \mathrm{Mat}_4(S_q^4)$ given by

$$e = \frac{1}{2} \begin{pmatrix} 1 + q^{-3/2}x_0 & q^{-1}(1+q)^{1/2}x_1 & (1+q)^{1/2}x_2 & 0 \\ q^{-1}(1+q)^{1/2}x_1^* & 1 - q^{-1/2}x_0 & 0 & (1+q)^{1/2}x_2 \\ (1+q)^{1/2}x_2^* & 0 & 1 - q^{-1/2}x_0 & -(1+q)^{1/2}x_1 \\ 0 & (1+q)^{1/2}x_2^* & -(1+q)^{1/2}x_1^* & 1 + q^{1/2}x_0 \end{pmatrix} \tag{3.12}$$

which is hermitian by construction, is also idempotent, $e^2 = e$, if and only if all the relations (3.9) and (3.11) which define S_q^4 are satisfied. For a generic value of the radius r the corresponding idempotent can be easily guessed from the previous expression and the analogue one (3.4) for the sphere S_q^2: one has simply to multiply the algebra generators in (3.12) by the constant $[(1 + q^3)/(1 + q)]^{1/2} r^{-1}$.

It is worth stressing the way the relations (3.3) for the sphere S_q^2 are incorporated in the ones (3.11) for the sphere S_q^4. This fact helped in constructing the projection (3.12) for S_q^4 by a suitable use of the projection (3.4) for S_q^2. Indeed, the projection $e_{(4)}$ in (3.12) is related to the analogous one $e_{(2)}$ in (3.4) by

$$e_{(4)} = \frac{1}{2} \begin{pmatrix} \mathbb{I}_2 + q^{-1} u_{(2)} & (1+q)^{1/2} x_2 \mathbb{I}_2 \\ (1+q)^{1/2} x_2^* \mathbb{I}_2 & \mathbb{I}_2 - u_{(2)} \end{pmatrix}, \tag{3.13}$$

with the hermitian isometry $u_{(2)}$ given by $u_{(2)} = 2e_{(2)} - 1$. By applying this 'inductive construction' one can obtain the projections for the higher-dimensional even spheres.

By using the one-dimensional representations (2.11) and a matrix trace Tr one finds that the projection (3.12) (and the corresponding vector bundle over S_q^4) is of rank 2,

$$\text{rank}(e) =: \tau_\lambda \circ \text{Tr}(e) = \tau_\lambda (2 + \tfrac{1}{2} q^{-3/2} (1-q)^2 x_0) = 2. \tag{3.14}$$

As it happens for the sphere S_q^2 described previously, in order to compute the topological charge of the bundle, we need a cyclic 0-cocycle, i.e. a trace τ^1, on the reduced algebra $\bar{S}_q^4 = S_q^4 / \mathbb{C}1$, which needs to be combined with the matrix trace. Now, the equivalence class of $\text{Tr}(e)$ in $\bar{S}_q^4$ is given by

$$[\text{Tr}(e)] = \tfrac{1}{2} q^{-3/2} (1-q)^2 x_0 \tag{3.15}$$

and we expect that the trace τ^1 should yield -1 on the previous expression. The reason to expect a value -1 for the topological charge is that in the classical limit $q = 1$, the bundle corresponding to the projection (3.12) is the instanton bundle over the classical sphere S^4 of topological charge (second Chern number) equal to -1; this is computed by integrating a 4-form on S^4 [19]. As for the sphere S_q^2, the trace τ^1 on S_q^4 will be then singular in the classical limit. As for now we have been unable to find such a singular trace on the sphere S_q^4.

3.3. Solitons on odd quantum Euclidean spheres

As already mentioned, odd quantum Euclidean spheres behave in a different manner than the even ones in so that the corresponding projectors are trivial (i.e. they correspond to trivial bundles). On S_q^3 this is a consequence of the fact that for the K-theory group one has $K_0(S_q^3) = \mathbb{Z}$ [21] and for higher values of N one expects a similar result.

3.3.1. The Euclidean sphere S_q^3

With a suitable rescaling of the generators the Euclidean sphere S_q^3 can be identified with the 'quantum sphere' $SU_q(2)$ of [28]. The present generators of the algebra are four

elements

$$x^i = (qx_2^*, x_1^*, x_1, x_2) \tag{3.16}$$

with commutation relations (2.8)

$$x_i x_j = q x_j x_i, \quad i < j, \qquad x_i^* x_j = q x_j x_i^*, \quad i \neq j, \qquad [x_1, x_1^*] = 0,$$
$$[x_2, x_2^*] = (1 - q^{-2}) x_1 x_1^*. \tag{3.17}$$

The central radial variable can be written as

$$r^2 = q^2 x_2 x_2^* + x_1 x_1^* + x_1^* x_1 + x_2^* x_2 = (1 + q^2)(x_2^* x_2 + x_1^* x_1)$$
$$= (1 + q^2)(x_2 x_2^* + q^{-2} x_1 x_1^*). \tag{3.18}$$

To alleviate the notation, we have chosen here the case $r^2 = (1 + q^2) \cdot 1$ so that the previous sphere relations reduce to

$$1 = x_2^* x_2 + x_1^* x_1 = x_2 x_2^* + q^{-2} x_1 x_1^*. \tag{3.19}$$

Then, straightforward computations yield that the element $e \in \mathrm{Mat}_4(S_q^3)$ given by

$$e = \frac{1}{2}
\begin{pmatrix}
1 & q^{-1} x_1 & x_2 & 0 \\
q^{-1} x_1^* & 1 & 0 & x_2 \\
x_2^* & 0 & 1 & -x_1 \\
0 & x_2^* & -x_1^* & 1
\end{pmatrix} \tag{3.20}$$

which is hermitian by construction, is also idempotent, $e^2 = e$, if and only if all the relations (3.17) and (3.19) which define S_q^3 are satisfied. For a generic value of the radius r the corresponding idempotent can be easily guessed from the previous expression: one has simply to multiply the algebra generators in (3.20) by the constant $(1 + q^2)^{1/2} r^{-1}$.

The projection (3.20) (and the corresponding vector bundle over S_q^3) is of rank 2,

$$\mathrm{rank}(e) = \tau_\lambda \circ \mathrm{Tr}(e) = \tau_\lambda(2) = 2, \tag{3.21}$$

with the representation τ_λ given in (2.11). By direct computation, one checks that the Chern character of the projection (3.20) vanished identically.

3.4. Solitons on the Euclidean sphere S_q^5

The generators of the algebra S_q^5 are the six elements

$$x^i = (q^2 x_3^*, q x_2^*, x_1^*, x_1, x_2, x_3), \tag{3.22}$$

with commutation relations (2.8) given by

$$x_i x_j = q x_j x_i, \quad i < j, \qquad x_i^* x_j = q x_j x_i^*, \quad i \neq j, \qquad [x_1, x_1^*] = 0,$$

$$[x_2, x_2^*] = (1 - q^{-2}) x_1 x_1^*, \qquad [x_3, x_3^*] = \frac{1 - q^{-2}}{1 + q^2} [q^2 x_2 x_2^* + x_1 x_1^* + x_1^* x_1 + x_2^* x_2]$$

$$= (1 - q^{-2})(x_2^* x_2 + x_1^* x_1) = (1 - q^{-2})(x_2 x_2^* + q^{-2} x_1 x_1^*). \tag{3.23}$$

It follows that the radial element can be written as

$$
\begin{aligned}
r^2 &= q^4 x_3 x_3^* + q^2 x_2 x_2^* + x_1 x_1^* + x_1^* x_1 + x_2^* x_2 + x_3^* x_3 \\
&= (1 + q^4)[x_3^* x_3 + x_2^* x_2 + x_1^* x_1] = (1 + q^4)[x_3 x_3^* + q^{-2}(x_2 x_2^* + q^{-2} x_1 x_1^*)].
\end{aligned}
$$
(3.24)

For the moment we consider the case $r^2 = (1 + q^4) \cdot 1$. Then, the previous sphere relations reduce to

$$
1 = x_3^* x_3 + x_2^* x_2 + x_1^* x_1 = x_3 x_3^* + q^{-2}(x_2 x_2^* + q^{-2} x_1 x_1^*).
$$
(3.25)

And again, straightforward computations yield that the element $e \in \mathrm{Mat}_8(S_q^5)$ given by

$$
e = \frac{1}{2}
\begin{pmatrix}
1 & q^{-2}x_1 & q^{-1}x_2 & 0 & x_3 & 0 & 0 & 0 \\
q^{-2}x_1^* & 1 & 0 & q^{-1}x_2 & 0 & x_3 & 0 & 0 \\
q^{-1}x_2^* & 0 & 1 & -q^{-1}x_1 & 0 & 0 & x_3 & 0 \\
0 & q^{-1}x_2^* & -q^{-1}x_1^* & 1 & 0 & 0 & 0 & x_3 \\
x_3^* & 0 & 0 & 0 & 1 & -q^{-1}x_1 & -x_2 & 0 \\
0 & x_3^* & 0 & 0 & -q^{-1}x_1^* & 1 & 0 & -x_2 \\
0 & 0 & x_3^* & 0 & -x_2^* & 0 & 1 & x_1 \\
0 & 0 & 0 & x_3^* & 0 & -x_2^* & x_1^* & 1
\end{pmatrix}
$$
(3.26)

which is hermitian by construction, is also idempotent, $e^2 = e$, if and only if all the relations (3.23) and (3.25) which define S_q^5 are satisfied. For a generic value of the radius r the corresponding idempotent can be easily guessed from the previous expression: one has simply to multiply the algebra generators in (3.26) by the constant $(1 + q^4)^{1/2} r^{-1}$.

Again, as for the even spheres, one should notice the way the relations (3.19) for the sphere S_q^3 are incorporated in the ones (3.25) for the sphere S_q^5. This fact helped in constructing the projection (3.26) for S_q^5 by a suitable use of the projection (3.20) for S_q^3. Indeed, the projection $e_{(5)}$ in (3.26) is related to the analogous one $e_{(3)}$ in (3.20) by

$$
e_{(5)} = \frac{1}{2}
\begin{pmatrix}
\mathbb{I}_4 + q^{-1}u_{(3)} & x_3 \mathbb{I}_4 \\
x_3^* \mathbb{I}_4 & \mathbb{I}_4 - u_{(3)}
\end{pmatrix},
$$
(3.27)

with the hermitian isometry $u_{(3)}$ given by $u_{(3)} = 2e_{(3)} - 1$. By applying this 'inductive construction' one can obtain the projections for the higher-dimensional odd spheres.

The projection (3.26) (and the corresponding vector bundle over S_q^5) is of rank 4,

$$
\mathrm{rank}(e) = \tau_\lambda \circ \mathrm{Tr}(e) = \tau_\lambda(4) = 4.
$$
(3.28)

As for the S_q^3 and the corresponding projection (3.20), the Chern character of the projection (3.26) vanished identically as well and the corresponding bundle is again trivial.

4. Normal and cotangent bundles

We shall now introduce a projection in a suitable matrix algebra which determines natural cotangent bundles over the spheres S_q^{N-1}. Consider then the following 'vector-valued function' on S_q^{N-1}:

$$\langle \perp | = \begin{cases} \dfrac{1}{r}(q^{-\rho_n}x_n, q^{-\rho_{n-1}}x_{n-1}, \ldots, q^{-\rho_1}x_1, x_0, x_1^*, \ldots, x_{n-1}^*, x_n^*) & \text{for } N=2n+1, \\[2mm] \dfrac{1}{r}(q^{-\rho_n}x_n, q^{-\rho_{n-1}}x_{n-1}, \ldots, q^{-\rho_1}x_1, x_1^*, \ldots, x_{n-1}^*, x_n^*) & \text{for } N=2n, \end{cases}$$

(4.1)

which is clearly normalized to 1: $\langle \perp | \perp \rangle = r^2/r^2 = 1$, from relation (2.9). Thus the matrix valued element $e_\perp \in \mathrm{Mat}_N(S_q^{N-1})$ given by

$$e_\perp = | \perp \rangle \langle \perp |$$

(4.2)

is a self-adjoint idempotent, i.e. $e_\perp^2 = e_\perp$, $e_\perp^* = e_\perp$. The corresponding finite projective module over the sphere S_q^{N-1} will be named the *normal bundle* over the sphere S_q^{N-1}. Furthermore, the self-adjoint idempotent

$$e_{\mathrm{Cot}} = \mathbb{I}_N - | \perp \rangle \langle \perp |$$

(4.3)

is a natural candidate for the *cotangent bundle* over the sphere S_q^{N-1}.

Again, the rank of the bundles is computed by combining the matrix trace with the one-dimensional representation in (2.11), $\mathrm{rank}(e) = \tau_\lambda \circ \mathrm{Tr}(e)$. Then, straightforward computations give that, as expected, the projector $e_\perp$ is of rank 1 while the projector e_{Cot} is of rank $N-1$,

$$\mathrm{rank}(e_\perp) = \tau_\lambda \circ \mathrm{Tr}(e_\perp) = \frac{1}{r^2}\tau_\lambda(q^{-2\rho_n}x_n^*x_n + \cdots + x_nx_n^*) = 1,$$

$$\mathrm{rank}(e_{\mathrm{Cot}}) = \tau_\lambda \circ \mathrm{Tr}(e_{\mathrm{Cot}}) = N - 1.$$

(4.4)

Relations of the cotangent projector (4.3) with differential calculi on the spheres S_q^{N-1} will be analyzed elsewhere.

5. Final remarks

We have presented a description of (the algebra of functions on) the quantum Euclidean spheres S_q^{N-1} by means of projections $e \in \mathrm{Mat}_{2^n}(S_q^{N-1})$, with $N = 2n$ or $N = 2n + 1$ which, in turn, can be regarded as bundles over the spheres S_q^{N-1}. As we shall report elsewhere, odd dimensional quantum Euclidean spheres are the same as the ones constructed in [26] from quantum unitary groups. Furthermore, at the C^*-algebra level even and odd quantum Euclidean spheres coincides with the corresponding spheres constructed in [17]. Additional and different spheres have been constructed in [1–3,9–12,25,27]. The monopole

presented here on the sphere S_q^2 coincides with the one constructed in [5] on the equator spheres of [24]. Our q-solitons and q-instantons on the spheres S_q^3, S_q^4 seems to be different from analogous objects recently found in [2,3,11,12,25]. It is also clearly different from the instanton constructed in [10]. The present Euclidean spheres are characterized by a 'homological dimension drop' which signals the fact that, contrary to the spheres S_θ^4 of [10], they are *not* noncommutative manifolds and their geometry cannot be a solution of homological equations like the ones in [10]. In [10], the spheres S_θ^4 were endowed with a noncommutative geometry via an even spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ where $\mathcal{A}$ is a noncommutative algebra with involution $*$ acting on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ (with grading given by γ) while D is a self-adjoint operator on $\mathcal{H}$ with compact resolvent and such that $[D, a]$ is bounded for any $a \in \mathcal{A}$ [7]. The operator D specify both the metric on the state space of $\mathcal{A}$ and the K-homology fundamental class [8]. The geometry for the spheres S_θ^4 is constructed by deforming the commutative triple $(C^\infty(S^4), \mathcal{H}, D, \gamma_5)$, where D is the Dirac operator on the Hilbert space $\mathcal{H}$ of square integrable spinors over S^4 and the grading is given by the 'fifth gamma matrix'. In fact, one has an isospectral deformation since both $\mathcal{H}$ and D are kept fixed, so that all spectral data of the geometry are unchanged, while the algebra and its representations are deformed.

An important problem that we leave for the future is the computation of topological charges, notably on the quantum Euclidean sphere S_q^4 and higher-dimensional even spheres. This is a difficult task since it involves the construction of Fredholm modules and their characters which, via the noncommutative index theorem of [7], pair integrally with the K_0 group. That the construction is difficult it is already evident from the analogous constructions [21,22] for the spheres S_q^2 and S_q^3. As a preliminary step for this construction we introduce some representations of the algebra of the quantum Euclidean sphere S_q^4 (similar representations can be constructed for any sphere and generalize known results for the quantum spheres S_q^2 and S_q^3).

Let us then consider again the algebra of S_q^4 which is specified by the commutation relations (3.9). And let $\mathcal{H}$ be an infinite-dimensional Hilbert space with orthonormal basis $\{\psi_{n,m}, n, m = 0, 1, 2, \ldots\}$. We have two representations (see also [14])

$$\pi(x_0)\psi_{n,m} = \pm r\sqrt{\frac{1+q}{1+q^3}}q^{n+1/2}q^{m+1}\psi_{n,m},$$

$$\pi(x_1)\psi_{n,m} = r\sqrt{\frac{1-q^{2(n+1)}}{1+q^3}}q^{m+1}\psi_{n+1,m}, \quad \pi(x_1^*)\psi_{n,m} = r\sqrt{\frac{1-q^{2n}}{1+q^3}}q^{m+1}\psi_{n-1,m},$$

$$\pi(x_2)\psi_{n,m} = r\sqrt{\frac{1-q^{2(m+1)}}{1+q^3}}\psi_{n,m+1}, \quad \pi(x_2^*)\psi_{n,m} = r\sqrt{\frac{1-q^{2m}}{1+q^3}}\psi_{n,m-1}, \quad (5.1)$$

with U a self-adjoint unitary operator on the Hilbert space $\mathcal{H}$.

We notice that for $q < 1$ any power of the operator $\pi(x_0)$ is a trace class operator while this is not the case for the operators $\pi(x_1)$, $\pi(x_1^*)$, $\pi(x_2)$ and $\pi(x_2^*)$ nor for any of their powers. Were this not the case there would be a contradiction with the algebra relations (3.9). A detailed analysis of the previous representations will be reported somewhere else.

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